Exercise 10

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = -\frac{1}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x + \int_0^x u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= -\frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= -\frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{-\frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x}_{u_0(x)} + \underbrace{\int_0^x u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= -\frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x \\
u_1(x) &= \int_0^x u_0(t)\,dt = \int_0^x \left(-\frac{1}{2}t - \frac{1}{4}\sin 2t + \cos^2 t\right) dt = -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}\sin 2x - \frac{1}{4}\sin^2 x \\
&\vdots
\end{aligned}$$

The noise terms, $\mp x/2$ and $\mp(\sin 2x)/4$, appear in both $u_0(x)$ and $u_1(x)$. Cancelling $-x/2$ and $-(\sin 2x)/4$ from $u_0(x)$ leaves $\cos^2 x$. Now we check to see whether $u(x) = \cos^2 x$ satisfies the integral equation.

$$\begin{aligned}
\cos^2 x &\stackrel{?}{=} -\frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + \int_0^x \cos^2 t\,dt \\
\cos^2 x &\stackrel{?}{=} -\cancel{\frac{1}{2}x} - \cancel{\frac{1}{4}\sin 2x} + \cos^2 x + \left(\cancel{\frac{1}{2}x} + \cancel{\frac{1}{4}\sin 2x}\right) \\
\cos^2 x &= \cos^2 x
\end{aligned}$$

Therefore,

$$u(x) = \cos^2 x.$$